The behavior of a graph

This narrative will hopefully clear up any confusion students have about using the first and second derivative to discover the behavior of a function.

Finding the critical numbers of a function

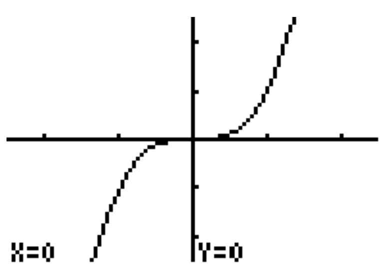
Remember that the first derivative tells us the slope of a graph at a certain value ‘c’ on the x axis. The slope is different along a non-linear function depending where on the graph your finding the slope (unlike a linear function where the slope is the same regardless where you are on the x axis).

What kind of line has a zero slope? Well, a horizontal line has a zero slope. So if I’m looking for a tangent line with a zero slope, what I’m really looking for is a horizontal tangent line. If a tangent line has a zero slope then that tangent line must either be sitting on top of a peak in my graph (a relative maximum in my graph) or must be touching a low point (or relative minimum in my graph).

These relative minimum and maximum values are called my critical points. To find them I take the first derivative of my function and set it equal to zero and solve for the variable (usually the x value). I then plug that x value into my original equation to get the y value and I now have a critical point.

It’s important to remember Fermat’s theorem, which says that IF my graph has a relative minimum or maximum, those min and max values will occur at the critical points. The converse of that statement is NOT true! Just because I have critical points does NOT mean my graph has a relative min or max.

For example, take the function $f(x) = x^3$. Taking the first derivative gives us $f'(x) = 3x^2$. If I set $f'(x)$ = 0 I get x =0 and y = 0. But let’s look at the graph of x^3:



You can see that (0,0) is not a minimum or a maximum. So you cannot assume that because you have your critical points you have a max or min in your graph. More on max and min later.

Finding the intervals of your graph and how the graph behaves

To find where your graph goes up and down, we need to look at the intervals across all values in the domain of $f(x)$. Why am I using that fancy word "domain"? Well, not every x value is in the domain of a function. Some functions are restricted, for example square root functions. Any values of x that will make my radicand (the stuff under the radical sign) negative are NOT in the domain of the function.

So we look at intervals that are to the left of the leftmost x value of the critical point, between each of the x value of the critical points, and to the right of the rightmost critical point. We then evaluate the first derivative at each of those intervals to see if the sign of the first derivative is positive or negative.

If it's negative, what does that tell you? Well, it means that the slope is negative. When is the slope negative? When the graph is going down, right? And when is the slope positive? When the slope is going up. So if we evaluate the first derivative at numbers between each interval, we find out whether the graph is going up or down. So if $f'(x)$ on an interval is > 0, then f is increasing function, and if $f'(x)$ is < 0 on an interval, then f is decreasing on that interval.

The concavity test

We wish to know if our graph is concave up or concave down. In other words, if the graph of the function is above the tangent line at a given point on your x axis, then the graph is concave up. If the graph is below the tangent line at a given point on your x axis, your graph is concave down. The second derivative of a function tells us whether a graph is concave up or concave down by evaluating the sign of our second derivative AT the critical point. So if the sign of the second derivative is positive, your graph is concave up. If the sign is negative, the graph is concave down.

Checking to see if you have a relative maximum or minimum.

So we know when a graph is going up or down. But how do you know if the critical points are a min or max of your function? That's when the second derivative test comes in handy. It says as follows: If the second derivative at the x value of the critical point is positive, then you have a relative minimum. Why? Well, remember that if $f''(x) > 0$ your graph is concave up and the tangent line is below the function f. So if at your critical point $f''(x) > 0$ then the graph is concave up at the point where you have a horizontal tangent line at $f'(x) = 0$. This MUST mean that you have a relative minimum at the critical point of your graph. Similarly, if $f''(x) < 0$ at the critical point, that means that your function is concave down at the point where you have a horizontal tangent line. This MUST mean that you have relative maximum. After all, how else would your graph be below the tangent line?

But suppose the $f''(x) = 0$, what does that tell you? Actually it doesn't tell you very much. You could have an absolute max, an absolute min or neither. That's when you need to do the first derivative test to the left and right of your critical points. Remember, the first derivative is the slope. If the first derivative at some value to the left of your critical point is negative and the first derivative at some value to the right of your critical point is positive, then you know that AT that critical point you have a minimum.

And if the first derivative at some value to the left of your critical point is positive and the first derivative at some value to the right of your critical point is negative, then you know that AT that critical point you have a maximum.

Inflection points

Your inflection point is the point on your graph where your function changes from concave up to concave down or concave down to concave up. That happens when your second derivative = zero. Why? Think about what the second derivative measures. It measures the change in your slope. In other words it measures when your functions changes concavity. To find that point, set $f''(x) = 0$ and solve for x.

Let's take the following example: $f(x) = 4x^3 + 3x^2 - 6x + 1$. We're going to want to do a few things:

1. Find the critical points of our graph.
2. Find the intervals of our graph using our critical points and determine where our graph is increasing or decreasing.
3. Using the second derivative, determine whether I have a relative max or relative min.
4. Using the second derivative, determine my inflection point

Let's start by finding the first and second derivative.

$f'(x) = 12x^2 + 6x - 6.$

$f''(x) = 24x + 6$

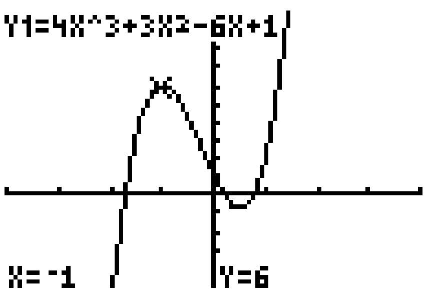
To find our critical points we set $f'(x) = 0$ so $12x^2 + 6x - 6 = 0$. Solving for x gives us $x = \frac{1}{2}, -1$. The intervals we use are as follows:

Interval	Sign of interval at $f'(x)$	Increasing/Decreasing
$-\infty \; to - 1$	(+)	INC
$-1 \; to \frac{1}{2}$	(-)	DEC
$\frac{1}{2} to + \infty$	(+)	INC

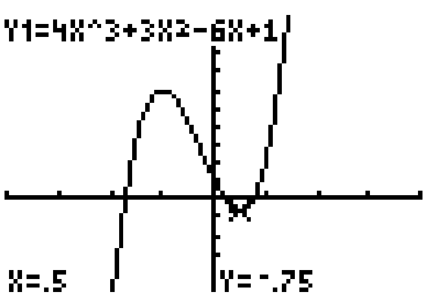
So our graph increases from $-\infty\ to -1$, decreases from $-1\ to \frac{1}{2}$ and increases again from $\frac{1}{2} to + \infty$. To check to see if our critical points are relative minimums or maximums, we evaluate $f''(x)$ at -1 and at $\frac{1}{2}$ and check the sign. So $f''\left(\frac{1}{2}\right) = 18$ which is positive, so at $x = \frac{1}{2}$ we have a relative min. But $f''(-1) = -18$ which is negative so we have a relative max at -1. To find the actual coordinate of the relative min and max, simply plug -1 into the $f(x)$ and we get 6, so our relative max is $(-1, 6)$. And we have a relative min at $(\frac{1}{2}, -\frac{3}{4})$.

Lastly we need to find the inflection point. To do that we set the $f''(x) = 0$ and solve, so $24x + 6 = 0$. Solving for x gives us $x = -\frac{1}{4}$. If we plug this into our $f(x)$ we get 2.625. Let's take a look at the graph.

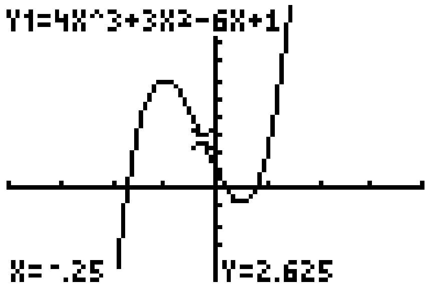
Now we'll look at the relative max at x = -1



Now we'll look at the relative min at $x = \frac{1}{2}$



And finally we'll take a look at the inflection point.


Y1=4X^3+3X²-6X+1
X=⁻.25
Y=2.625