MATH 150
FINAL EXAM α
Dec. 14, 2009

5 $\underline{\text{BTS}}$

Closed book, closed notes. SHOW ALL WORK to get full credit. Calculators are allowed.

Name and section: ____________________

Instructor's name: ____________________

1. Multiple Choice. Evaluate the following limits:

(a) $\lim_{x\to 3}(3x^3+5x-1) =$ $3(27)+15-1 = 81+15-1 = 95$

(i) 41 (ii) 95 (iii) 23 (iv) 0 (v) None of these

(b) $\lim_{x\to -3}\frac{x^2-9}{x+3} =$ $\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}} = (x-3) = -6$

(i) 0 (ii) ∞ (iii) -6 (iv) 6 (v) None of these

(c) $\lim_{x\to 0}\frac{\sin(2x)\cos(7x)}{x^2} =$ $\frac{\frac{\sin(2x)\cos(7x)}{2x}}{\frac{x^2}{2x}} = \frac{2\cos(7x)}{x}$ = undefined

(i) 1 (ii) 0 (iii) 2 (iv) 14 (v) None of these

(d) $\lim_{x\to\infty}\sqrt{x^2+x}-\sqrt{x^2+1} =$ $\frac{x^2+x-x^2-1}{\sqrt{x^2+x}+\sqrt{x^2+1}} = \frac{\frac{x-1}{x}}{\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{1}{x^2}}} = \frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{1}{x^2}}} = \frac{1}{2}$

(i) Does not exist (ii) 1 (iii) 0 (iv) 1/2 (v) None of these

2. Multiple choice. Compute the derivatives of the following functions:

(a) If $f(x) = 2x^4 - 3x^2 - 9$ then $f'(x) =$

(i) $2x^3 - 3x$ (ii) $8x^3 - 6x$ (iii) $8x^3 - 6x - 9$ (iv) $\frac{2}{5}x^5 - x^3 - 9x$ (v) None of these

$F'(x) = 8x^3 - 6x$

(b) If $g(x) = \tan^3(\sec x)$ then $g'(x) =$

(i) $3\cancel{\tan}^2(\sec x)$ (ii) $3\tan^2(\sec x \tan x)$ (iii) $(3\tan^2(\sec x))(\sec x \tan x)$
(iv) $(3\tan^2(\sec x))\sec^2(\sec x)\sec x \tan x$ (v) None of these

$g'(x) = 3\tan^2(\sec x) * \sec^2(\sec x) * \sec x \tan x$

(c) If $y = \frac{x^2+1}{x^3-1}$ then $\frac{dy}{dx} = \frac{3x^2(x^2+1) - 2x(x^3-1)}{(x^3-1)^2} =$

(i) $\frac{-x^4 - 3x^2 - 2x}{(x^3-1)^2}$ (ii) $\frac{2x}{3x^2}$ (iii) $\frac{(x^3-1)(2x) - (3x^2)(x^2+1)}{x^3-1}$
(iv) $\frac{(x^3-1)(2x) - (3x^2)(x^2+1)}{(x^3-1)^3}$ (v) None of these

(d) If $h(x) = (x^2+4)(\sin(x^2+2))$ then $h'(x) =$

(i) $\cancel{(2x)(\cos(x^2+2))}$ (ii) $\cancel{(4x^2)(\sin(x^2+2))}$ (iii) $(4x^2)(\cos(x^2+2))$
(iv) $(2x)[\sin(x^2+2) + (x^2+4)\cos(x^2+2)]$ (v) None of these

$h'(x) = (x^2+4)(\cos(x^2+2)) * 2x + 2x(\sin(x^2+2))$

$2x(x^2+4)(\cos(x^2+2)) + 2x(\sin(x^2+2))$

3. An cannon ball is shot straight upward from the ground - the height at time t is given by $s(t) = -16t^2 + 64t$.

(a) Find the velocity when $t = 1/4$.

$s'(t) = -32t + 64$

$s'(\frac{1}{4}) = -8 + 64 = \boxed{56}$

(b) How high does the cannon ball go?

$-32t + 64 = 0$

$-32t = -64$

$t = 2$

$-16(2)^2 + 64(2) =$

$-64 + 128 = \boxed{64}$

(c) How many seconds after its release does the cannon ball strike the ground?

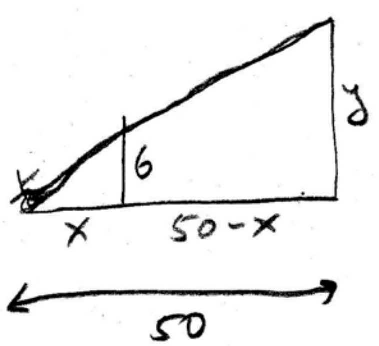
$-16t^2 + 64t = 0$

$t(-16t + 64) = 0$

$t = 0 \quad t = +4$

$\boxed{\text{AT } t = 4}$

4. A man 6 ft. tall is walking toward a building at a rate of 5 ft./sec. If there is a light *on the ground* 50 ft. from the building, how fast is the man's shadow on the building growing shorter when he is 20 ft. from the building?



$\frac{6}{x} = \frac{y}{50 - x}$

$\frac{300 - 6x}{x} = y$

$300x^{-1} - 6 = y$

$\frac{dx}{dt} - 300x^{-2} = \frac{dy}{dt}$

$(5) - 300\left(\frac{1}{400}\right) = \frac{dy}{dt}$

$\frac{20}{4} - \frac{3}{4} = \frac{dy}{dt}$

$\boxed{\frac{dy}{dt} = \frac{17}{4}}$

5. You need to make a rectangular box with square base and no top, having a total surface area of 64 sq. in. What are the dimensions of the box giving maximal volume? Explain why your answer is a maximum.

$$64 = 2xh + 2xh + x^2$$

$$64 = 4xh + x^2$$

$$\frac{64 - x^2}{4x} = h$$

$$V = x^2 \frac{(64 - x^2)}{4x} = \frac{-x^3 + 64x}{4}$$

$$V' = -\frac{3}{4}x^2 + 16 \qquad -\frac{3}{4}x^2 + 16 = 0 \Rightarrow -\frac{3}{4}x^2 = -16$$

$$x^2 = \frac{64}{3} \Rightarrow x = \pm\frac{8\sqrt{3}}{3} \approx \pm\boxed{4.62}$$ Length width

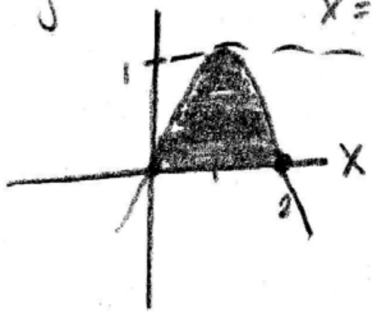
$$\frac{64 - \frac{8\sqrt{3}}{3}}{\frac{32\sqrt{3}}{3}} = \frac{\frac{192 - 8\sqrt{3}}{3}}{\frac{32\sqrt{3}}{3}} = \frac{192 - 8\sqrt{3}}{32\sqrt{3}} = \frac{24 - \sqrt{3}}{4\sqrt{3}} \approx \frac{24\sqrt{3} - 3}{12} = \frac{8\sqrt{3} - 1}{4} \approx \boxed{3.21}$$

height

6. Consider the region bounded by $y = 2x - x^2$ and the x-axis.

(a) Find the area of this region.

$y' = -2x+2 \qquad -2x+2=0 \qquad x=1$



$x(2-x)=0$

$x=0 \quad x=2$

$$\int_0^2 2x-x^2\,dx = x^2-\frac{x^3}{3}\Big|_0^2 = 4-\frac{8}{3} = \frac{12-8}{3} = \boxed{\frac{4}{3}}$$

(b) Consider the solid obtained by rotating this region around the x-axis. Calculate the volume of this solid.

$$\int_0^2 \pi(2x-x^2)^2\,dx = \pi\int_0^2 4x^2-2x^3-2x^3+x^4\,dx = \int_0^2 4x^2-4x^3+x^4\,dx$$

$$= \frac{4}{3}x^3-x^4+\frac{x^5}{5}\Big|_0^2 = \frac{4}{3}(8)-(16)+\frac{32}{5} = \frac{160}{15}-\frac{240}{15}+\frac{96}{15} = \frac{16\pi}{15} \approx$$

(c) Consider the solid obtained by rotating this region around the line $y = 1$. Calculate the volume of this solid.

$$\int_0^2 2\pi(1)^2 - 2\pi(1-2x+x^2)^2\,dx$$

7. Consider this function, along with its derivative and its second derivative (which you do not have to verify):

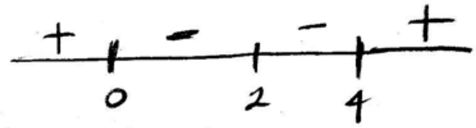
$$y = \frac{x^2 - 2x + 4}{x - 2}; \qquad y' = \frac{x(x-4)}{(x-2)^2}; \qquad y'' = \frac{8}{(x-2)^3}.$$

(a) What are the critical points (if any)?

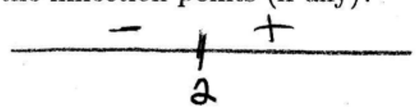
$\frac{x(x-4)}{(x-2)^2} = 0$

$x = 0,\ x = 4,\ x = 2$

(b) On what intervals is the function increasing and on what intervals is the function decreasing?



(c) What are the inflection points (if any)?

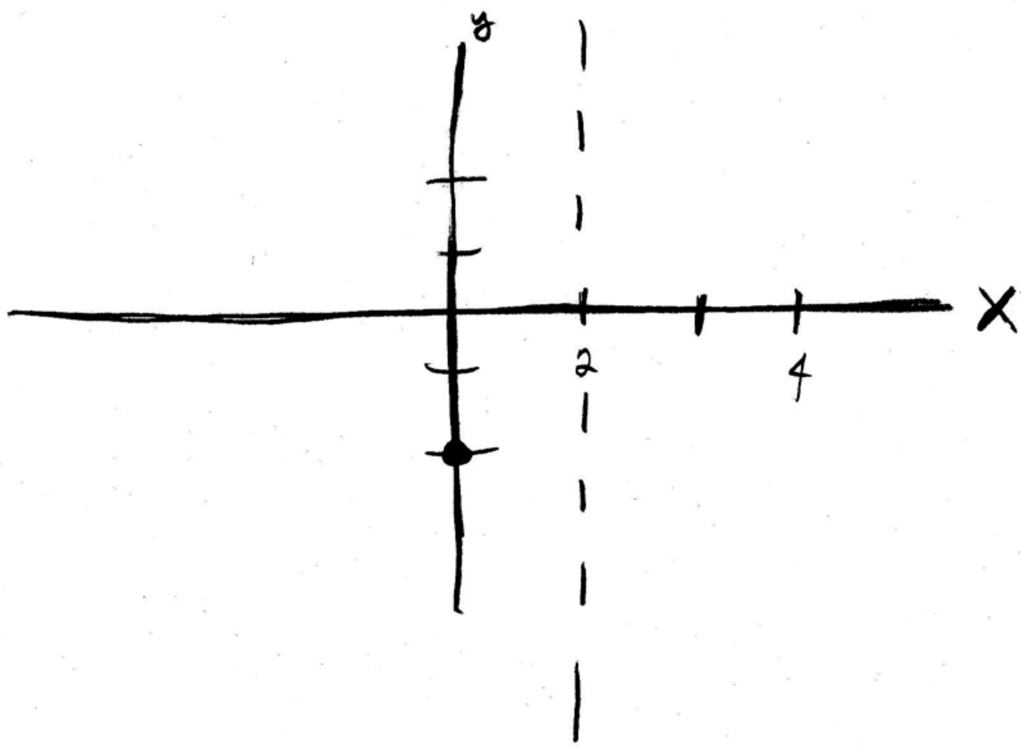


$x = 2$

(d) On what intervals is the function concave up and on what intervals is the function concave down?

$(-\infty, 2]$ down $\qquad [2, \infty)$ up

(e) Using this information, sketch the graph, indicating any maxima and minima and asymptotes.



8. Evaluate the following integrals using the *fundamental theorem of calculus* (Note: some of these are definite integrals, some are indefinite)

(a) $\int_0^2 (2x^3 + x^2 + 1)\,dx$

$$\frac{x^4}{2} + \frac{x^3}{3} + x\Big|_0^2 = \frac{16}{2} + \frac{8}{3} + 2 = \frac{48}{6} + \frac{16}{6} + \frac{12}{6}$$

$$\frac{76}{6} = \frac{38}{3} = 12.6\overline{6}$$

(b) $\int x\sqrt{x^2 - 1}\,dx$

let $u = x^2 - 1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\boxed{\frac{1}{\cancel{2}}\int u^{1/2}\,du = \frac{2u^{3/2}}{3} = \frac{\cancel{2}(x^2-1)^{3/2}}{3} + C}$$

(c) $\int_0^{\pi/4} \tan^2\theta \sec^2\theta\,d\theta$

let $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$d\theta = \frac{du}{\sec^2\theta}$

$$\boxed{\int_0^{\pi/4} u^2\,du = \frac{\tan^3\theta}{3}\Big|_0^{\pi/4} = \frac{1}{3}}$$

(d) $\int x^3\cos(x^4 + 6)\,dx$

let $u = x^4 + 6$

$du = 4x^3\,dx$

$dx = \frac{du}{4x^3}$

$$\boxed{\frac{1}{4}\int \cos u\,du = -\frac{\sin(x^4+6)}{4} + C}$$