## 7-5 SOLVING EQUATIONS INVOLVING EXPONENTS

We know that to raise a power to a power, we multiply exponents. Therefore, for positive values of $x$ and non-zero integer values of $a$:

$$(x^a)^{\frac{1}{a}} = x^{a\left(\frac{1}{a}\right)} = x^1 = x \qquad \left(x^{\frac{1}{a}}\right)^a = x^{\frac{1}{a}(a)} = x^1 = x$$

We can use this relationship to solve for $x$ in an equation such as $x^{\frac{2}{3}} = 25$. To solve for $x$, we need to raise $x^{\frac{2}{3}}$ to the power that is the reciprocal of the exponent $\frac{2}{3}$. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

$$\begin{aligned} x^{\frac{2}{3}} &= 25 \\ \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} &= 25^{\frac{3}{2}} \\ x^1 &= 25^{\frac{3}{2}} \\ x &= 25^{\frac{3}{2}} \end{aligned}$$

Note that $25^{\frac{3}{2}}$ means $\left(25^{\frac{1}{2}}\right)^3$, that is, the cube of the square root of 25.

$$x = \left(\sqrt{25}\right)^3 = 5^3 = 125$$

### EXAMPLE 1

Solve each equation and check: **a.** $2a^{-3} - 1 = 15$ **b.** $2\sqrt[3]{x^5} + 1 = 487$

***Solution***

| *How to Proceed* | **a.** $2a^{-3} - 1 = 15$ | **b.** $2\sqrt[3]{x^5} + 1 = 487$ |
|---|---|---|
| (1) Write the equation with only the variable term on one side of the equation: | $2a^{-3} = 16$ | $2\sqrt[3]{x^5} = 487$ |
| (2) Divide both sides of the equation by the coefficient of the variable term: | $a^{-3} = 8$ | $x^{\frac{5}{3}} = 243$ |
| (3) Raise both sides of the equation to the power that is the reciprocal of the exponent of the variable: | $(a^{-3})^{-\frac{1}{3}} = 8^{-\frac{1}{3}}$<br>$a = 8^{-\frac{1}{3}}$ | $(x^{\frac{5}{3}})^{\frac{3}{5}} = 243^{\frac{3}{5}}$<br>$x = 243^{\frac{3}{5}}$ |
| (4) Simplify the right side of the equation: | $a = \frac{1}{8^{\frac{1}{3}}}$<br>$= \frac{1}{\sqrt[3]{8}}$<br>$= \frac{1}{2}$ | $x = 243^{\frac{3}{5}}$<br>$= \left(\sqrt[5]{243}\right)^3$<br>$= 3^3$<br>$= 27$ |

(5) *Check* the solution:

**a.**
$$\begin{aligned} 2a^{-3} - 1 &= 15 \\ 2\left(\tfrac{1}{2}\right)^{-3} - 1 &\stackrel{?}{=} 15 \\ 2\,(2)^3 - 1 &\stackrel{?}{=} 15 \\ 2(8) - 1 &\stackrel{?}{=} 15 \\ 16 - 1 &\stackrel{?}{=} 15 \\ 15 &= 15 \checkmark \end{aligned}$$

**b.**
$$\begin{aligned} 2\sqrt[3]{x^5} + 1 &= 487 \\ 2\sqrt[3]{27^5} + 1 &\stackrel{?}{=} 487 \\ 2\sqrt[3]{(3^3)^5} + 1 &\stackrel{?}{=} 487 \\ 2\sqrt[3]{3^{15}} + 1 &\stackrel{?}{=} 486 \\ 2(3^5) + 1 &\stackrel{?}{=} 487 \\ 487 &= 487 \checkmark \end{aligned}$$

*Answers* **a.** $a = \frac{1}{2}$ **b.** $x = 27$ ❑

## Exercises

### Writing About Mathematics

**1.** Ethan said that to solve the equation $(x + 3)^{\frac{1}{2}} = 5$, the first step should be to square both sides of the equation. Do you agree with Ethan? Explain why or why not.

**2.** Chloe changed the equation $a^{-2} = 36$ to the equation $\frac{1}{a^2} = \frac{1}{36}$ and then took the square root of each side. Will Chloe's solution be correct? Explain why or why not.

### Developing Skills

In 3–17 solve each equation and check.

**3.** $x^{\frac{1}{3}} = 4$ **4.** $a^{\frac{1}{5}} = 2$ **5.** $x^{\frac{2}{3}} = 9$

**6.** $b^{\frac{3}{2}} = 8$ **7.** $x^{-2} = 9$ **8.** $b^{-5} = \frac{1}{32}$

**9.** $2y^{-1} = 12$ **10.** $9a^{-\frac{3}{4}} = \frac{1}{3}$ **11.** $5x^{\frac{3}{4}} = 40$

**12.** $5x^{\frac{1}{2}} + 7 = 22$ **13.** $14 - 4b^{\frac{1}{3}} = 2$ **14.** $(2x)^{\frac{1}{2}} + 3 = 15$

**15.** $3a^3 = 81$ **16.** $x^5 = 3{,}125$ **17.** $z^{\frac{1}{2}} = \sqrt{81}$

In 18–23, solve for the variable in each equation. Express the solution to the nearest hundredth.

**18.** $x^{-3} = 24$ **19.** $y^{\frac{3}{5}} = 6$ **20.** $a^{-\frac{3}{4}} = 0.85$

**21.** $3z^3 + 2 = 27$ **22.** $5 + b^5 = 56$ **23.** $(3w)^9 + 2 = 81$

**24.** Solve for $x$ and check: $\frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}} = 10$. Use the rule for the division of powers with like bases to simplify the left side of the equation.

### Applying Skills

**25.** Show that if the area of one face of a cube is $B$, the volume of the cube is $B^{\frac{3}{2}}$.

**26.** If the area of one face of a cube is $B$ and the volume of the cube is $V$, express $B$ in terms of $V$.

# 7-6 SOLVING EXPONENTIAL EQUATIONS

## Solving Exponential Equations With the Same Base

An **exponential equation** is an equation that contains a power with a variable exponent. For example, $2^{2x} = 8$ and $5^{x-1} = 0.04$ are exponential equations.

An exponential function $y = b^x$ is a one-to-one function since it is increasing for $b > 1$ and decreasing for $0 < b < 1$. Let $y_1 = b^{x_1}$ and $y_2 = b^{x_2}$. If $y_1 = y_2$, then $b^{x_1} = b^{x_2}$ and $x_1 = x_2$.

▶ **In general, if $b^p = b^q$, then $p = q$.**

We can use this fact to solve exponential equations that have the same base.

**EXAMPLE 1**

Solve and check: $3^x = 3^{2x-2}$

***Solution*** Since the bases are equal, the exponents must be equal.

$$\begin{aligned} 3^x &= 3^{2x-2} \\ x &= 2x - 2 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Check

$$\begin{aligned} 3^x &= 3^{2x-2} \\ 3^2 &\stackrel{?}{=} 3^{2(2)-2} \\ 3^2 &= 3^2 \checkmark \end{aligned}$$

*Answer* $x = 2$ ■

## Solving Exponential Equations With Different Bases

How do we solve exponential equations such as $2^{2x} = 8$ or $5^{x-1} = 0.04$? One approach is, if possible, to write each term as a power of the same base. For example:

$$\begin{aligned} 2^{2x} &= 8 \\ 2^{2x} &= 2^3 \\ 2x &= 3 \\ x &= \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} 5^{x-1} &= 0.04 \\ 5^{x-1} &= \tfrac{4}{100} \\ 5^{x-1} &= \tfrac{1}{25} \\ 5^{x-1} &= \tfrac{1}{5^2} \\ 5^{x-1} &= 5^{-2} \\ x - 1 &= -2 \\ x &= -1 \end{aligned}$$

**EXAMPLE 2**

Solve and check: $4^a = 8^{a+1}$

***Solution*** The bases, 4 and 8, can each be written as a power of 2: $4 = 2^2$, $8 = 2^3$.

$$\begin{aligned} 4^a &= 8^{a+1} \\ (2^2)^a &= (2^3)^{a+1} \\ 2^{2a} &= 2^{3a+3} \\ 2a &= 3a + 3 \\ -a &= 3 \\ a &= -3 \end{aligned}$$

Check

$$\begin{aligned} 4^a &= 8^{a+1} \\ 4^{-3} &\stackrel{?}{=} 8^{-3+1} \\ 4^{-3} &\stackrel{?}{=} 8^{-2} \\ \frac{1}{4^3} &\stackrel{?}{=} \frac{1}{8^2} \\ \frac{1}{64} &= \frac{1}{64} \checkmark \end{aligned}$$

*Answer* $a = -3$

**EXAMPLE 3**

Solve and check: $3 + 7^{x-1} = 10$

***Solution*** Add $-3$ to each side of the equation to isolate the power.

$$\begin{aligned} 3 + 7^{x-1} &= 10 \\ 7^{x-1} &= 7 \\ x - 1 &= 1 \\ x &= 2 \end{aligned}$$

Check

$$\begin{aligned} 3 + 7^{x-1} &= 10 \\ 3 + 7^{2-1} &\stackrel{?}{=} 10 \\ 3 + 7^1 &\stackrel{?}{=} 10 \\ 10 &= 10 \checkmark \end{aligned}$$

*Answer* $x = 2$

## Exercises

### Writing About Mathematics

**1.** What value of $a$ makes the equation $6^a = 1$ true? Justify your answer.

**2.** Explain why the equation $3^a = 5^{a-1}$ cannot be solved using the procedure used in this section.

### Developing Skills

In 3–14, write each number as a power.

**3.** 9 **4.** 27 **5.** 25 **6.** 49

**7.** 1,000 **8.** 32 **9.** $\frac{1}{8}$ **10.** $\frac{1}{216}$

**11.** 0.001 **12.** 0.125 **13.** 0.81 **14.** 0.16

In 15–38, solve each equation and check.

**15.** $2^x = 16$

**16.** $3^x = 27$

**17.** $5^x = \frac{1}{5}$

**18.** $7^x = \frac{1}{49}$

**19.** $4^{x+2} = 4^{2x}$

**20.** $3^{x+1} = 3^{2x+3}$

**21.** $6^{3x} = 6^{x-1}$

**22.** $3^{x+2} = 9^x$

**23.** $25^x = 5^{x+3}$

**24.** $49^x = 7^{3x+1}$

**25.** $2^{2x+1} = 16^x$

**26.** $9^{x-1} = 27^x$

**27.** $100^x = 1{,}000^{x-1}$

**28.** $125^{x-1} = 25^x$

**29.** $6^{2-x} = \left(\frac{1}{36}\right)^2$

**30.** $\left(\frac{1}{4}\right)^x = 8^{1-x}$

**31.** $\left(\frac{1}{3}\right)^x = 9^{1-x}$

**32.** $(0.01)^{2x} = 100^{2-x}$

**33.** $(0.25)^{x-2} = 4^x$

**34.** $5^{x-1} = (0.04)^{2x}$

**35.** $4^x + 7 = 15$

**36.** $5 + 7^x = 6$

**37.** $e^{2x+2} = e^{x-1}$

**38.** $3^{x^2+2} = 3^6$